Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Lecture 10: Operations on Multiple Random Variables

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Operations on Multiple Random Variables

1. Expected Value of a Function of Random Variables

2. Joint Moments

3. Joint Characteristic Functions

4. Variance for Joint Distributions (Covariance)

5. Conditional Expectation, Variance, and Moments

6. Examples

1. Expected Value of a Function of Random Variables

- ✓ The **expected value** of a function of random variables **X** and **Y** is given by

$$\overline{g} = E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

- ✓ For **N** random variables **X_1 , X_2 , ….. X_N**

$$\overline{g} = E[g(X_1,\cdots,X_N)] = \int_{-\infty}^{+\infty}\cdots\cdots\int_{-\infty}^{+\infty} g(x_1,\cdots,x_N)\, f_{X_1,\cdots,X_N}(x_1,\cdots,x_N)\,dx_1\cdots dx_N$$

2. Joint Moments

Joint Moments about the Origin (*Product Moments*):

- ✓ The **r_{th}** and **s_{th}** **product moment** about the **origin** of the random variables **X** and **Y**, denoted by $\mu'_{r,s}$ is the expected value of X^rY^s symbolically

$$\mu'_{r,s} = E(X^rY^s) = \sum_x \sum_y x^r y^s \cdot f(x,y)$$

For **r** $= 0, 1, 2, \ldots$ and **s** $= 0, 1, 2, \ldots$ when **X** and **Y** are **discrete**,

And

$$\mu'_{r,s} = E(X^r Y^s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^r y^s \cdot f(x, y) dx dy$$

When **X** and **Y** are **continuous**.

- ✓ The **joint moments** of the random variables **X** and **Y** about the **origin** are defined by

$$m_{nk} = E[X^n Y^k] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^n y^k \; f_{X,Y}(x, y) dx dy$$

- ○ The **sum n+k** called the **order of the moments.**

Clearly:

$$m_{n0} = E[X^n] \quad \text{are the moments of } X$$
$$m_{0k} = E[Y^k] \quad \text{are the moments of } Y$$

- ✓ The **first order** joint moments:

$$m_{10} = E[X] = \overline{X} \quad \text{the mean value of } X$$
$$m_{01} = E[Y] = \overline{Y} \quad \text{the mean value of } Y$$

✓The **second order** joint moments:

$$m_{20} = E[X^2] \quad \text{the mean square value of } X$$
$$m_{02} = E[Y^2] \quad \text{the mean square value of } Y$$
$$m_{11} = E[XY] \quad \text{the correlation of } X \text{ and } Y$$

✓The **correlation** of **X** and **Y** is important to later work:

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

a) If

$$R_{XY} = E[X]E[Y]$$

Then **X** and **Y** said to be **uncorrelated**

b) **Statistical independence** of **X** and **Y** is sufficient to guarantee they are **uncorrelated** but the converse is not necessarily true in general.

c) If

$$R_{XY} = 0$$

Then **X** and **Y** called **orthogonal** random variables.

✓ **Example:**

Let **X** be a random variable that has a mean value $E[X] = 3$ and variance $\sigma_X^2 = 2$. Another random variable is defined by $Y = -6X + 22$. Find the mean value of **Y**, the variance of **Y** and the correlation of **X** and **Y**.

○ **Solution:**

$$\sigma_X^2 = E[X^2] - \overline{X}^2 \quad \Rightarrow \quad E[X^2] = \sigma_X^2 + \overline{X}^2 = 2 + 3^2 = 11$$

$$E[Y] = E[-6X + 22] = -6E[X] + 22 = 4$$

$$E[Y^2] = E[(-6X + 22)^2] = E[36X^2 - 264X + 484]$$

$$= 36E[X^2] - 264E[X] + 484 = 88$$

$$\sigma_Y^2 = E[Y^2] - \overline{Y}^2 = 72$$

$$R_{XY} = E[XY] = E[-6X^2 + 22X] = -6E[X^2] + 22E[X]$$

$$= -6(11) + 22(3) = 0$$

Since $R_{XY} = 0$, X and Y are orthogonal

$R_{XY} \neq E[X]E[Y]$, X and Y are not uncorrelat ed

Joint Central Moments (*Product Moments about the mean*):

- ✓ The r_{th} and s_{th} product moment about the means of the random variables **X** and **Y**, denoted by $\mu_{r,s}$ is the expected value of $(X-\mu_X)^r(Y-\mu_Y)^s$ symbolically

$$\mu_{r,s} = E[(X-\mu_X)^r(Y-\mu_Y)^s] = \sum_x \sum_y (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)$$

For $r = 0, 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ when **X** and **Y** are **discrete**,

And

$$\mu_{r,s} = E[(X-\mu_X)^r(Y-\mu_Y)^s] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)\,dx\,dy$$

When **X** and **Y** are **continuous**.

- ✓ The **joint central moments** of the random variables **X** and **Y** are defined by

$$\mu_{nk} = E[(X-\overline{X})^n(Y-\overline{Y})^k] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\overline{X})^n(y-\overline{Y})^k \, f_{X,Y}(x,y)\,dx\,dy$$

  - The **sum n+k** called the **order of the moments**.

✓ The **first order central** moments:

$$\mu_{10} = E[X - \overline{X}] = 0$$
$$\mu_{01} = E[Y - \overline{Y}] = 0$$

✓ The **second order central** moments:

$$\mu_{20} = E[(X - \overline{X})^2] = \sigma_X^2$$
$$\mu_{02} = E[(Y - \overline{Y})^2] = \sigma_Y^2$$
$$\mu_{11} = E[(X - \overline{X})(Y - \overline{Y})] = C_{XY}$$

✓ The second order moment μ_{11} is called the **covariance** of **X** and **Y** and is given by

$$C_{XY} = \mu_{11} = E[(X - \overline{X})(Y - \overline{Y})]$$
$$= R_{XY} - E[X]E[Y]$$

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{XY}}{\sigma_X \sigma_Y}$$

is called the correlatio n coefficien t

a) If **X** and **Y** are either **independent** or **uncorrelated** then

$$C_{XY} = 0$$

b) If **X** and **Y** are **orthogonal** then

$$C_{XY} = -E[X]E[Y]$$

✓ **Example:** From the previous example

$$\overline{X} = 3 \qquad \sigma_X^2 = 2 \qquad Y = -6X + 22$$
$$\overline{Y} = 4 \qquad R_{XY} = 0$$

Find the **covariance** of **X** and **Y**.

○ **Solution:**

$$C_{XY} = R_{XY} - \overline{X}\,\overline{Y} = -12$$

Note that

$$C_{XY} = -\overline{X}\,\overline{Y}, \quad \text{because } X \text{ and } Y \text{ are orthogonal}$$

3. Joint Characteristic Functions

✓The **joint characteristic function** of two random variables **X** and **Y** is defined by

$$\Phi_{X,Y}(\omega_1,\omega_2)=E[e^{j\omega_1 X+j\omega_2 Y}]$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{j\omega_1 x+j\omega_2 y}\, f_{X,Y}(x,y)dxdy$$

✓The **joint moments** can be found as follows

$$m_{nk}=(-j)^{n+k}\left.\frac{\partial^{n+k}\Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^n\,\partial\omega_2^k}\right|_{\omega_1=0,\omega_2=0}$$

$\Phi_X(\omega_1)=\Phi_{X,Y}(\omega_1,0)$ and $\Phi_Y(\omega_2)=\Phi_{X,Y}(0,\omega_2)$

are the marginal characteri stic functions

✓ **Example:**

Two random variables **X** and **Y** have the joint characteristic function

$$\Phi_{X,Y}(\omega_1,\omega_2) = e^{-2\omega_1^2 - 8\omega_2^2}$$

$$\text{Find } \ \overline{X},\ \overline{Y}, R_{XY} \text{ and } C_{XY}$$

○ **Solution:**

$$\overline{X} = m_{10} = (-j)^{1+0} \frac{\partial^1 \Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^1\,\partial\omega_2^0}\bigg|_{\omega_1=0,\omega_2=0} = -j(-4\omega_1)e^{-2\omega_1^2-8\omega_2^2}\Big|_{\omega_1=0,\omega_2=0} = 0$$

$$\overline{Y} = m_{01} = (-j)^{0+1} \frac{\partial^1 \Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^0\,\partial\omega_2^1}\bigg|_{\omega_1=0,\omega_2=0} = -j(-16\omega_2)e^{-2\omega_1^2-8\omega_2^2}\Big|_{\omega_1=0,\omega_2=0} = 0$$

$$R_{XY} = m_{11} = (-j)^{1+1} \frac{\partial^2 \Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^1\,\partial\omega_2^1}\bigg|_{\omega_1=0,\omega_2=0} = -(-4\omega_1)(-16\omega_2)e^{-2\omega_1^2-8\omega_2^2}\Big|_{\omega_1=0,\omega_2=0} = 0$$

$$C_{XY} = R_{XY} - \overline{X}\,\overline{Y} = 0 \quad \Rightarrow \quad X \text{ and } Y \text{ are uncorrelat ed}$$

4. Variance for Joint Distributions (Covariance)

✓ If ***X*** and ***Y*** are two continuous random variables having joint density function ***f*** **(*x*, *y*)**, then

- The **means**, or **expectations**, of ***X*** and ***Y*** are

$$\mu_X = E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x, y)\,dx\,dy, \qquad \mu_Y = E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x, y)\,dx\,dy$$

- The **variances** are

$$\sigma_X^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_X)^2 f(x, y)\,dx\,dy$$

$$\sigma_Y^2 = E[(Y - \mu_Y)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - \mu_Y)^2 f(x, y)\,dx\,dy$$

- Another quantity that arises in the case of two variables ***X*** and ***Y*** is the **covariance** defined by

$$\sigma_{XY} = \mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

  - In terms of the **joint density function** ***f*** **(*x*, *y*),** we have

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f(x, y)\,dx\,dy$$

✓ For two discrete random variables

$$\mu_X = \sum_x \sum_y x f(x, y) \qquad \mu_Y = \sum_x \sum_y y f(x, y)$$

$$\sigma_{XY} = \sum_x \sum_y (x - \mu_X)(y - \mu_Y) f(x, y)$$

Some important theorems on covariance

✓ **Theorem 1:**

$$\sigma_{XY} = E(XY) - E(X)E(Y) = E(XY) - \mu_X \mu_Y$$

✓ **Theorem 2:** If X and Y are **independent** random variables, then

$$\sigma_{XY} = \text{Cov}(X, Y) = 0$$

✓ **Theorem 3**

$$\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2\,\text{Cov}(X, Y)$$

Or

$$\sigma_{X \pm Y}^2 = \sigma_X^2 + \sigma_Y^2 \pm 2\sigma_{XY}$$

✓ **Theorem 4**

$$|\sigma_{XY}| \le \sigma_X \sigma_Y$$

Correlation Coefficient

✓ If **X** and **Y** are completely dependent, for example, when **X =Y**, then

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = \sigma_X \sigma_Y$$

- From this we are led to a ***measure of the dependence*** of the variables **X** and **Y** given by

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- We call ρ the ***correlation coefficient***, or ***coefficient of correlation***, $-1 \leq \rho \leq 1$.
- In the case where $\rho = 0$ (i.e., the covariance is zero), we call the variables **X** and **Y** ***uncorrelated***

5. Conditional Expectation, Variance, and Moments

✓ If **X** and **Y** have joint density function **f (x, y)**, then we can define the ***conditional expectation***, or ***conditional mean***, of **Y** given **X** by

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y f(y \mid x)\, dy$$

✓ We note the following properties

$$1.\ E(Y|X = x) = E(Y) \text{ when } X \text{ and } Y \text{ are independent.}$$

$$2.\ E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_1(x)\, dx.$$

✓ In a similar manner, we can define the ***conditional variance*** of Y given **X** as

$$E[(Y - \mu_2)^2 | X = x] = \int_{-\infty}^{\infty} (y - \mu_2)^2 f(y|x)\, dy$$

Where

$$\mu_2 = E(Y|X = x)$$

✓ Also we can define the ***rth conditional moment*** of *Y* about any value *a* given *X* as

$$E[(Y - a)^r | X = x] = \int_{-\infty}^{\infty} (y - a)^r f(y|x)\, dy$$

6. Examples

✓ **Example 1:** The joint and marginal probabilities of ***X*** and ***Y***, are recorded as follows:

		$x=0$	$x=1$	$x=2$	
y	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{7}{12}$
	1	$\frac{2}{9}$	$\frac{1}{6}$		$\frac{7}{18}$
	2	$\frac{1}{36}$			$\frac{1}{36}$
		$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{12}$	

Find the **covariance** of ***X*** and ***Y***.

- **Solution**
  - Referring to the **joint probabilities** given here, we get

$$\mu'_{1,1} = E(XY)$$
$$= 0\cdot 0\cdot \frac{1}{6} + 0\cdot 1\cdot \frac{2}{9} + 0\cdot 2\cdot \frac{1}{36} + 1\cdot 0\cdot \frac{1}{3} + 1\cdot 1\cdot \frac{1}{6} + 2\cdot 0\cdot \frac{1}{12}$$
$$= \frac{1}{6}$$

  - and using the **marginal** probabilities, we get

$$\mu_X = E(X) = 0\cdot \frac{5}{12} + 1\cdot \frac{1}{2} + 2\cdot \frac{1}{12} = \frac{2}{3}$$

- and

$$\mu_Y = E(Y) = 0 \cdot \frac{7}{12} + 1 \cdot \frac{7}{18} + 2 \cdot \frac{1}{36} = \frac{4}{9}$$

- It follows that

$$\sigma_{XY} = \frac{1}{6} - \frac{2}{3} \cdot \frac{4}{9} = -\frac{7}{54}$$

✓ **Example 2:** Find the **covariance** of the random variables whose joint probability density is given by

$$f(x, y) = \begin{cases} 2 & \text{for } x > 0, y > 0, x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

- **Solution**

$$\mu_X = \int_0^1 \int_0^{1-x} 2x \, dy \, dx = \frac{1}{3}$$

$$\mu_Y = \int_0^1 \int_0^{1-x} 2y \, dy \, dx = \frac{1}{3}$$

and

$$\sigma'_{1,1} = \int_0^1 \int_0^{1-x} 2xy \, dy \, dx = \frac{1}{12}$$

It follows that

$$\sigma_{XY} = \frac{1}{12} - \frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{36}$$

✓ **Example 3:** If the joint probability distribution of ***X*** and ***Y*** is given by the following table. Show that the covariance, of the two random variables, is zero even though they are **not independent**.

$y \backslash x$	-1	0	1	
-1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{3}$
0	0	0	0	0
1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

- **Solution**

$$\mu_X = (-1)\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = 0$$

$$\mu_Y = (-1)\cdot\frac{2}{3} + 0\cdot 0 + 1\cdot\frac{1}{3} = -\frac{1}{3}$$

$$\mu'_{1,1} = (-1)(-1)\cdot\frac{1}{6} + 0(-1)\cdot\frac{1}{3} + 1(-1)\cdot\frac{1}{6} + (-1)1\cdot\frac{1}{6} + 1\cdot 1\cdot\frac{1}{6} = 0$$

Thus,

$$\sigma_{XY} = 0 - 0(-\tfrac{1}{3}) = 0$$

✓ **Example 4:** Suppose that the random variables X and Y have a **joint mass function** given by

$$f(x,y) = c(2x+y)$$

Where

$$0 \le x \le 2, 0 \le y \le 3$$

Find

(a) $E(X)$, **(b)** $E(Y)$, **(c)** $E(XY)$, **(d)** $E(X^2)$, **(e)** $E(Y^2)$,
(f) $Var\ (X)$, **(g)** $Var\ (Y)$, **(h)** $Cov\ (X,Y)$, **(i)** ρ

- **Solution**

X \ Y	0	1	2	3	Totals ↓
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$5c$	$6c$	$7c$	$22c$
Totals →	$6c$	$9c$	$12c$	$15c$	$42c$

(a)

$$E(X) = \sum_x \sum_y x f(x, y) = \sum_x x \left[\sum_y f(x, y) \right]$$

$$= (0)(6c) + (1)(14c) + (2)(22c) = 58c = \frac{58}{42} = \frac{29}{21}$$

(b)

$$E(Y) = \sum_x \sum_y y f(x, y) = \sum_y y \left[\sum_x f(x, y) \right]$$

$$= (0)(6c) + (1)(9c) + (2)(12c) + (3)(15c) = 78c = \frac{78}{42} = \frac{13}{7}$$

(c)

$$E(XY) = \sum_x \sum_y xy f(x, y)$$

$$\begin{aligned} &= (0)(0)(0) + (0)(1)(c) + (0)(2)(2c) + (0)(3)(3c) \\ &\quad + (1)(0)(2c) + (1)(1)(3c) + (1)(2)(4c) + (1)(3)(5c) \\ &\quad + (2)(0)(4c) + (2)(1)(5c) + (2)(2)(6c) + (2)(3)(7c) \\ &= 102c = \frac{102}{42} = \frac{17}{7} \end{aligned}$$

(d)

$$E(X^2) = \sum_x \sum_y x^2 f(x, y) = \sum_x x^2 \left[\sum_y f(x, y) \right]$$

$$= (0)^2(6c) + (1)^2(14c) + (2)^2(22c) = 102c = \frac{102}{42} = \frac{17}{7}$$

(e) $$E(Y^2) = \sum_x \sum_y y^2 f(x,y) = \sum_y y^2 \left[\sum_x f(x,y) \right]$$

$$= (0)^2(6c) + (1)^2(9c) + (2)^2(12c) + (3)^2(15c) = 192c = \frac{192}{42} = \frac{32}{7}$$

(f) $$\sigma_X^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{17}{7} - \left(\frac{29}{21}\right)^2 = \frac{230}{441}$$

(g) $$\sigma_Y^2 = \text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{32}{7} - \left(\frac{13}{7}\right)^2 = \frac{55}{49}$$

(h) $$\sigma_{XY} = \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{17}{7} - \left(\frac{29}{21}\right)\left(\frac{13}{7}\right) = -\frac{20}{147}$$

(i) $$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-20/147}{\sqrt{230/441}\sqrt{55/49}} = \frac{-20}{\sqrt{230}\sqrt{55}} = -0.2103 \text{ approx.}$$

✓ **Example 5:** Suppose that the random variables X and Y have a **joint density function** given by

$$f(x,y) = \begin{cases} \frac{1}{210}(2x+y) & 2 < x < 6,\ 0 < y < 5 \\ 0 & Otherwise \end{cases}$$

Find

(a) $E(X)$, **(b)** $E(Y)$, **(c)** $E(XY)$, **(d)** $E(X^2)$, **(e)** $E(Y^2)$,
(f) $Var\ (X)$, **(g)** $Var\ (Y)$, **(h)** $Cov\ (X,Y)$, **(i)** ρ

- **Solution**

(a) $$E(X) = \frac{1}{210}\int_{x=2}^{6}\int_{y=0}^{5}(x)(2x+y)\,dx\,dy = \frac{268}{63}$$

(b) $$E(Y) = \frac{1}{210}\int_{x=2}^{6}\int_{y=0}^{5}(y)(2x+y)\,dx\,dy = \frac{170}{63}$$

(c) $$E(XY) = \frac{1}{210}\int_{x=2}^{6}\int_{y=0}^{5}(xy)(2x+y)\,dx\,dy = \frac{80}{7}$$

(d) $$E(X^2) = \frac{1}{210}\int_{x=2}^{6}\int_{y=0}^{5}(x^2)(2x+y)\,dx\,dy = \frac{1220}{63}$$

(e) $$E(Y^2) = \frac{1}{210}\int_{x=2}^{6}\int_{y=0}^{5}(y^2)(2x+y)\,dx\,dy = \frac{1175}{126}$$

(f) $$\sigma_X^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{1220}{63} - \left(\frac{268}{63}\right)^2 = \frac{5036}{3969}$$

(g) $$\sigma_Y^2 = \mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{1175}{126} - \left(\frac{170}{63}\right)^2 = \frac{16{,}225}{7938}$$

(h) $$\sigma_{XY} = \mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{80}{7} - \left(\frac{268}{63}\right)\left(\frac{170}{63}\right) = -\frac{200}{3969}$$

(i) $$\rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{-200/3969}{\sqrt{5036/3969}\sqrt{16{,}225/7938}} = \frac{-200}{\sqrt{2518}\sqrt{16{,}225}} = -0.03129 \text{ approx.}$$

✓ **Example 6:** Proof the following theorem

$$\sigma_{XY} = \mu'_{1,\,1} - \mu_X \mu_Y$$

- **Solution**

Proof By definition the covariance of X and Y and using the various theorems about expected values, we can write

$$\begin{aligned}
\sigma_{XY} &= \mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] \\
&= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\
&= E(XY) - \mu_X E(Y) - \mu_Y E(X) + E(\mu_X \mu_Y) \\
&= E(XY) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y \\
&= E(XY) - \mu_X \mu_Y \\
&= E(XY) - E(X)E(Y)
\end{aligned}$$

✓ **Example 7:** Proof the following theorem

If X and Y are **independent**, then $E(XY) = E(X) \cdot E(Y)$ and $\sigma_{XY} = 0$.

- **Solution**

Proof For the discrete case we have, by definition,

$$E(XY) = \sum_x \sum_y xy \cdot f(x, y)$$

Since ***X*** and ***Y*** are **independent**, we can write $\boldsymbol{f(x, y) = g(x) \cdot h(y)}$, where $\boldsymbol{g(x)}$ and $\boldsymbol{h(y)}$ are the values of the marginal distributions of ***X*** and ***Y***, and we get

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy \cdot g(x)h(y) \\ &= \left[\sum_x x \cdot g(x)\right]\left[\sum_y y \cdot h(y)\right] \\ &= E(X) \cdot E(Y) \end{aligned}$$

Hence,

$$\begin{aligned} \sigma_{XY} &= \mu'_{1,1} - \mu_X \mu_Y \\ &= E(X) \cdot E(Y) - E(X) \cdot E(Y) \\ &= 0 \end{aligned}$$

✓ **Theorem (Remark)**

If

$X1, X2, \ldots, Xn$ are **independent**

Then

$$\boldsymbol{E(X1X2 \ldots Xn) = E(X1) \cdot E(X2) \cdot \ldots \cdot E(Xn)}$$

This is a **generalization** of the first part of above theorem.